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How to estimate the Value at Risk under incomplete information

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ABSTRACT

A key problem in financial and actuarial research, and particularly in the field of risk management, is the choice of models so as to avoid systematic biases in the measurement of risk. An alternative consists of relaxing the assumption that the probability distribution is completely known, leading to interval estimates instead of point estimates. In the present contribution, we show how this is possible for the Value at Risk, by fixing only a small number of parameters of the underlying probability distribution. We start by deriving bounds on tail probabilities, and we show how a conversion leads to bounds for the Value at Risk. It will turn out that with a maximum of three given parameters, the best estimates are always realized in the case of a unimodal random variable for which two moments and the mode are given. It will also be shown that a lognormal model results in estimates for the Value at Risk that are much closer to the upper bound than to the lower bound.

1. Introduction

One of the most widely applied risk measures nowadays is the Value at Risk, which is used to quantify large losses related to the probabilities of their occurrence. This concept was introduced at the end of the 1980s, and since then, it has become increasingly popular in the financial world, especially as measurement for the market risk (see [1,2]). Although there are some shortcomings, the importance of this Value at Risk can be illustrated by referring to the "Basel II" regulations about the risk management of financial institutions, as well as to the regulations of the US Securities and Exchange Commission. Both institutions explicitly mention the concept of Value at Risk as one of the recommended or compulsory risk measures (see e.g. [3,4]).

The Value at Risk is defined as the amount of loss such that the probability of running a loss this large or even larger over a certain period of time, is limited. For example, if the Value at Risk at 99% is equal to 1 million euros in two weeks, this means that the probability of being confronted with a loss of 1 million euros or more in two weeks is limited to 1%. Another way to explain this is that we can be 99% confident that we will not lose more than 1 million euros in two weeks. In a more formal way, we can define the Value at Risk of a variable X for any percentile p as

$$\mathrm{VaR}_p(X) = \inf\{t \in \mathbb{R} \mid \mathrm{Prob}(X \geqslant t) \leqslant 1 - p\}, \quad p \in (0, 1) \tag{1}$$

where by convention $\inf\{\emptyset\} = \infty$.

If we know the probability distribution of the losses, the Value at Risk can be derived immediately for all percentiles. The knowledge of this probability distribution, however, constitutes the difficult point in the reasoning. In most practical applications and methods it is assumed that the underlying distribution is normal or lognormal, and the Value at Risk is calculated starting from this hypothesis. Although both models have their strong points and perform well in many cases,

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they are not perfect descriptions of reality, and as such, they can cause serious biases. For example, most financial time series exhibit fatter tails than the commonly used models. This has been indicated by several authors, see e.g. [5–9]. One possible solution to this problem is to refine the models, so that specific characteristics of the real process can be incorporated into the model. A second possibility – the one we want to contribute to – is to leave the hypothesis of working with a complete model, and to try to resolve the problem by relaxing the assumption that the probability distribution is completely known.

Now, if the exact model for the distribution of losses were known, the Value at Risk could be calculated as one single value for each choice of the probability p. On the other hand, if we do not know the exact distribution, but just some parameters, e.g. the mean and variance, possibly based on historical data, it is no longer possible to find a single outcome for the Value at Risk. In that case, we can derive a range of possible results or an interval estimate, and especially the upper and lower bounds for the Value at Risk. More particularly, the real Value at Risk of a certain percentile is restricted to the range between this upper and lower bound, regardless of the exact underlying distribution for the losses under investigation.

The method we use in order to derive these upper and lower bounds is somewhat technical. In a first step, we apply a method we developed earlier (see [10,11]) leading to general restrictions on tail probabilities, or on probabilities of reaching high values [12,13]. Afterwards, we transform these results into upper and lower bounds for the Value at Risk (see Section 2 for more details).

The classical choices when fixing only some parameters of the underlying distributions are the mean, variance and possibly the skewness of the model. For technical reasons, we use in this contribution the non-central moments, which we denote by $\mu_k = \mathrm{E}[X^k]$, where X is the variable under investigation. The equivalence between the central and non-central moments is well-known and straightforward, with $\mu := \mathrm{E}[X] = \mu_1$, $\sigma^2 := \mathrm{E}[(X-\mu)^2] = \mu_2 - \mu_1^2$ and $\gamma_1 := \mathrm{E}[(X-\mu)^3]/\sigma^3 = (\mu_3 - 3\mu_1\mu_2 + 2\mu_1^3)/(\mu_2 - \mu_1^2)^{3/2}$.

In addition to the situation where we have information about the successive moments of the investigated (unknown) model, we look at the situation where we know the mode of the model. This means that in this particular case we assume that the underlying and unknown distribution is unimodal. This is undoubtedly a meaningful hypothesis: indeed, the most popular and most widely-used models for the evolution of capital, interest rates etc. are always unimodal, e.g. normal, lognormal and gamma models.

In the present contribution, we present exact upper and lower bounds assuming three parameters of the underlying distribution are fixed. In particular, we assume that we have sufficient information about three successive moments, or about two moments and the mode if the distribution is assumed to be unimodal. These general restrictions will then be valid for all possible distributions (continuous, discrete or hybrid) with the same values for the particular parameters. For the results corresponding to situations with fewer than three parameters, references will be provided.

The paper is organized as follows. We describe our method in Section 2. We then present the results for the bounds on the tail probabilities in Section 3, and for the bounds on the Value at Risk in Section 4. We present numerical examples in Section 5 and the conclusion in Section 6.

2. Method

As explained in the introduction, the calculation of the Value at Risk of a random variable can be linked to the calculation of tail probabilities (see Eq. (1)), and thus it seems reasonable to couple the problem of finding bounds for both quantities.

In this section, we explain how to derive bounds on tail probabilities and on the Value at Risk. For the first problem, or the estimation of tail probabilities, we make use of a method introduced in [14], and further refined in [10]. We then show how these results can be transformed into restrictions on the Value at Risk.

2.1. Bounds for tail probabilities

The problem consists of marking out the feasible range of all possible values for the tail probability of a variable X, which can be written as $\mathrm{Prob}(X \geqslant t) = \mathrm{E}\left[1_{[t,+\infty)}(X)\right]$, where $1_{[t,+\infty)}$ as usual denotes the indicator function on the interval $[t, +\infty)$.

If the variable X has range $[0, b]$, with $b \in \mathbb{R}_0^+$, this means that we have to determine

$$\sup_{F\in\mathcal{B}} \int_0^b 1_{[t,+\infty)}(x)\mathrm{d}F(x) \quad \text{and} \quad \inf_{F\in\mathcal{B}} \int_0^b 1_{[t,+\infty)}(x)\mathrm{d}F(x), \tag{2}$$

where $\mathcal{B}$ is the class of all possible distribution functions with domain $[0, b]$ and with given moments and/or mode.

If we fix successive moments of the distribution, we can follow the approach as in [14], with the following reasoning. If $P(x)$ is a polynomial of degree 3 or less, the value of the integral $\int_0^b P(x)\mathrm{d}F(x)$ only depends on the first three moments of F. If $\mathcal{B}$ is the class of all distribution functions with domain $[0, b]$ and with the first three moments fixed, this value is the same for each distribution $F \in \mathcal{B}$. This means that the problem of finding the supremum can be reduced to the problem of finding such a polynomial $P(x)$ greater than $1_{[t,+\infty)}(x)$ on $[0, b]$ and such that for some distribution $F \in \mathcal{B}$ we have $\int_0^b P(x)\mathrm{d}F(x) = \int_0^b 1_{[t,+\infty)}(x)\mathrm{d}F(x)$. When looking for the infimum, this polynomial should be smaller than $1_{[t,+\infty)}(x)$ on $[0, b]$.

If the variable X is unimodal and in addition to some moments, the mode is also given, it is possible to include this information in the calculation of the upper and lower bounds, and to transform the problem of (one or) two moments with mode into a problem of (one or) two moments without mode. Following [15], this can be done through the following Lemma, which refers back to Khinchin's characterization of unimodality (see [16]).

Definition 2.1. For any continuous real function $g : [0, b] \to \mathbb{R} : x \mapsto g(x)$ and for any real number m between 0 and b, the Khinchin transform of g with respect to m is the real function $h : [0, b] \to \mathbb{R} : x \mapsto h(x)$ defined by $h(x) = \frac{1}{x-m} \int_m^x g(z)\mathrm{d}z$.

Lemma 2.2. *If a unimodal variable X has range $[0, b]$, mode m and moments μ_1 and μ_2, then a random variable Y exists with the same range $[0, b]$ and with moments $\nu_1 = 2\mu_1 - m$ and $\nu_2 = 3\mu_2 - 2m\mu_1$, such that for any function $g : x \mapsto g(x)$ the following equality holds:*

$$\mathrm{E}[g(X)] = \mathrm{E}[h(Y)]$$

where h is the Khinchin transform of g with respect to the mode m.

Following [10], we can use a Khinchin transform (see Definition 2.1) to derive bounds for tail probabilities with two moments and mode of the underlying distribution fixed. This means that the problem (2) changes to

$$\sup_{G\in\mathcal{D}} \int_0^b h(x)\mathrm{d}G(x) \quad \text{and} \quad \inf_{G\in\mathcal{D}} \int_0^b h(x)\mathrm{d}G(x), \tag{3}$$

with $\mathcal{D}$ the class of all distribution functions with domain $[0, b]$ and with two moments transformed as in Lemma 2.2, with h the Khinchin transform of the indicator function.

For distributions, we construct point distributions belonging to $\mathcal{B}$ or $\mathcal{D}$; for polynomial P we choose the polynomial which is nowhere smaller or nowhere larger then the indicator function (or the Khinchin transform) and which equals this function in the mass points of a point distribution such that $\int_0^b P(x)\mathrm{d}F(x) = \int_0^b 1_{[t,+\infty)}(x)\mathrm{d}F(x)$ or $\int_0^b P(x)\mathrm{d}G(x) = \int_0^b h(x)\mathrm{d}G(x)$. In so doing, the method results in upper and lower bounds that can be reached within the class $\mathcal{B}$ or $\mathcal{D}$, which illustrates the usefulness of point distributions. In Appendix A we demonstrate how in particular it is possible to construct such two and three point distributions.

The approach outlined in this section can be applied to many other problems. Indeed, many distribution-driven quantities which can be written as an expectation of a random variable, or $Q(X) = \mathrm{E}[f(X)]$, can be treated in the same way. In each case, the problem consists of finding the right polynomials and the right point distributions as indicated earlier. As an example, we can refer to [17], where this methodology was adapted to construct upper and lower bounds for option prices. In a risk-neutral world, an option price can be written by means of an expectation as $Q(X)$, and thus it is possible to derive general restrictions for option prices by fixing only a few parameters of the distribution of the price process.

2.2. Conditions on the parameters

From elementary conditions on distribution functions, we can derive some essential conditions on the parameters used throughout this paper, in order to guarantee the existence of a distribution. These are summarized in the second Lemma.

Lemma 2.3. *If a variable X has range $[0, b]$, and moments μ_1, μ_2 and μ_3, then these parameters have to satisfy the following inequalities:*

$$\begin{aligned} &\bullet\ b\mu_1 \geqslant \mu_2 \quad \text{and} \quad b\mu_2 \geqslant \mu_3 \\ &\bullet\ \mu_2 \geqslant \mu_1^2 \quad \text{and} \quad \mu_3\mu_1 \geqslant \mu_2^2. \end{aligned} \tag{4}$$

If a unimodal variable X has range $[0, b]$, mode m and moments μ_1 and μ_2, then these parameters also have to satisfy the following inequalities:

$$\begin{aligned} &\bullet\ 2\mu_1 \geqslant m \\ &\bullet\ 2(b+m)\mu_1 - bm \geqslant 3\mu_2 \geqslant m^2 + 4\mu_1^2 - 2m\mu_1. \end{aligned} \tag{5}$$

Note that equalities can only hold in the case of some discrete distributions.

The first set of inequalities in (4) is due to the fact that the variable under investigation is assumed to be limited to the domain $[0, b]$, whereas the second set follows from the fact that both $(X - \mu_1)^2$ and $X \cdot (X - \mu_2/\mu_1)^2$ are non-negative. The inequalities in (5) are the result of an application of the earlier requirements to the moments of the transformed variable as defined in Lemma 2.2.

Finally, if the distribution is unimodal with a known mode, we restrict ourselves to the situation where $\mu_1 \geqslant m$ for technical reasons. This is the most interesting case, since we then have a distribution with a right tail.

Table 1
Upper and lower bounds for $\text{Prob}(X \geqslant t)$ (Theorem 3.1).

t	$f_{low}^{(1)}(t)$	$f_{upp}^{(1)}(t)$
$0 \leqslant t \leqslant \kappa_1$	$\frac{\mu_2-\mu_1(t+b)+tb}{(t-s_t)(b-s_t)} + \frac{\mu_2-\mu_1(s_t+t)+s_t t}{(b-s_t)(b-t)}$	1
$\kappa_1 < t \leqslant \frac{b\mu_2-\mu_3}{b\mu_1-\mu_2}$	$\frac{(\mu_2-t\mu_1)^3}{(\mu_3-t\mu_2)(\mu_3-2t\mu_2+t^2\mu_1)}$	$\frac{\mu_1}{t} - \frac{(\mu_2-t\mu_1)^2}{t(\mu_3-t\mu_2)}$
$\frac{b\mu_2-\mu_3}{b\mu_1-\mu_2} < t \leqslant \kappa_2$	$\frac{\mu_2-\mu_1(s_t+t)+s_t t}{(b-s_t)(b-t)}$	$\frac{\mu_2-\mu_1(s_t+t)+s_t t}{(b-s_t)(b-t)} + \frac{\mu_2-\mu_1(s_t+b)+s_t b}{(s_t-t)(b-t)}$
$\kappa_2 < t \leqslant b$	0	$\frac{\mu_3\mu_1-\mu_2^2}{t(\mu_3-2t\mu_2+t^2\mu_1)}$

2.3. Bounds for the Value at Risk

The definition of the Value at Risk, see Eq. (1), immediately proves a strong connection between this Value at Risk and tail probabilities. Making use of this link, it seems logical to convert the bounds for tail probabilities into bounds for the Value at Risk.

Indeed, since the tail probability $\text{Prob}(X \geqslant t)$ is a non-increasing function of t, it follows from $1-q \leqslant \text{Prob}(X \geqslant t) \leqslant 1-p$ that $\text{VaR}_p(X) \leqslant t$ and $\text{VaR}_q(X) \geqslant t$.

Relying on the previous formulae, we can formulate the two following Lemmas, forming the basis for the calculations of the transformations.

Lemma 2.4. *If for $t_1 < t < t_2$ we have $\text{Prob}(X \geqslant t) \geqslant f_{low}(t)$, then for $p_1 < p < p_2$ with $p_1 = 1-f_{low}(t_1)$ and $p_2 = 1-f_{low}(t_2)$ it is true that $\text{VaR}_p(X) \geqslant f_{low}^{-1}(1-p)$.*

Lemma 2.5. *If for $\bar{t}_1 < t < \bar{t}_2$ we have $\text{Prob}(X \geqslant t) \leqslant f_{upp}(t)$, then for $\bar{p}_1 < p < \bar{p}_2$ with $\bar{p}_1 = 1-f_{upp}(\bar{t}_1)$ and $\bar{p}_2 = 1-f_{upp}(\bar{t}_2)$ it is true that $\text{VaR}_p(X) \leqslant f_{upp}^{-1}(1-p)$.*

This shows how lower and upper bounds for the tail probabilities can be converted into lower and upper bounds for the Value at Risk.

For explicit expressions for the functions f_{low} and f_{upp} we refer to Theorems 3.1 and 3.2 from Section 3. Note that the nature of the different functions f_{low} and f_{upp} when limited to the corresponding domain $[t_{i-1}, t_i]$ implies that the inverse functions in both Lemmas are well defined.

As there are no singularities with respect to the end point b, the results can be extended into results for variables with range $\mathbb{R}^+$ by having b tend to infinity, see also [14].

3. General restrictions for tail probabilities

In this section, we present bounds for tail probabilities when three parameters are fixed, i.e. three successive moments or two moments and the mode. For cases up to two parameters, the results were published originally in [12]. They can also be found in a more recent technical report [13], where all the bounds for the situation $\mu_1 \geqslant m$ are summarized in a uniform way. The proof of the new result in Theorem 3.1 can be found in Appendix B, as for Theorem 3.2, we refer to [12].

3.1. Knowledge of the first three moments

Define $\kappa_1 \leq \kappa_2$ as the real roots of the equation

$$(\mu_2 - \mu_1^2)\kappa^2 + (\mu_1\mu_2 - \mu_3)\kappa + (\mu_1\mu_3 - \mu_2^2) = 0$$

and for each $t \in [0, b]$ define s_t as

$$s_t = \frac{(b\mu_2 - \mu_3) - t(b\mu_1 - \mu_2)}{(b\mu_1 - \mu_2) - t(b - \mu_1)}.$$

Note that with these definitions and under the conditions of Lemma 2.3, it is always true that $\kappa_1 \leqslant \frac{b\mu_2-\mu_3}{b\mu_1-\mu_2} \leqslant \kappa_2$, which is important for the appropriateness of the entries for t in the result of the next theorem.

Theorem 3.1. *Consider a random variable X with support $[0, b]$, for which the first three moments are given by μ_1, μ_2 and μ_3. The tail probability is then bounded as*

$$f_{low}^{(1)}(t) \leqslant \text{Prob}(X \geqslant t) \leqslant f_{upp}^{(1)}(t)$$

with $f_{low}^{(1)}$ and $f_{upp}^{(1)}$ as in Table 1.

Table 2
Upper and lower bounds for Prob($X \geqslant t$) (Theorem 3.2 — first case).

t	$f_{low}^{(2)}(t)$
$0 < t \leqslant \frac{m\nu_2}{\nu_2+2m\nu_1}$	$\frac{\nu_1^2 t+\nu_2(m-t)}{m\nu_2}$
$\frac{m\nu_2}{\nu_2+2m\nu_1} < t \leqslant m$	$\frac{m-t}{m-x_t}\frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1 x_t+x_t^2}+\frac{(\nu_1-x_t)^2}{\nu_2-2\nu_1 x_t+x_t^2}$
$m < t \leqslant b'$	$\frac{(\nu_1-t)^2}{\nu_2-\nu_1^2+(\nu_1-m)(\nu_1-t)}$
$b' < t \leqslant \frac{\nu_2}{\nu_1}$	$\frac{\nu_2-t\nu_1}{b(b-m)}$
$\frac{\nu_2}{\nu_1} < t \leqslant b$	0

t	$f_{upp}^{(2)}(t)$
$0 < t \leqslant m$	1
$m < t \leqslant \frac{bb'^2}{b(2b'-m)+(b'-m)^2}$	$\frac{(\nu_1-b')(b-t)}{(b-b')(b-m)}+\frac{(b-\nu_1)(b'-t)}{(b-b')(b'-m)}$
$\frac{bb'^2}{b(2b'-m)+(b'-m)^2} < t \leqslant \frac{bo'^2}{b(2o'-m)+(o'-m)^2}$	$\frac{(b\nu_1-\nu_2)(g_t-t)}{g_t(b-g_t)(g_t-m)}+\frac{(\nu_2-\nu_1 g_t)(b-t)}{b(b-g_t)(b-m)}$
$\frac{bo'^2}{b(2o'-m)+(o'-m)^2} < t \leqslant \frac{o'(2o'-m)}{3o'-2m}$	$\frac{\nu_1^2(\nu_2-t\nu_1)}{\nu_2(\nu_2-m\nu_1)}$
$\frac{o'(2o'-m)}{3o'-2m} < t \leqslant \frac{b(2b-m)-b'm}{3b-2m-b'}$	$\frac{y_t-t}{y_t-m}\cdot\frac{\nu_2-\nu_1^2}{y_t^2-2\nu_1 y_t+\nu_2}$
$\frac{b(2b-m)-b'm}{3b-2m-b'} < t \leqslant b$	$\frac{b-t}{b-m}\cdot\frac{\nu_2-\nu_1^2}{b^2-2\nu_1 b+\nu_2}$

3.2. Knowledge of the first two moments and the mode

Consider ν_1 and ν_2 as in Lemma 2.2, define $o' = \frac{\nu_2}{\nu_1}$ and $b' = \frac{b\nu_1-\nu_2}{b-\nu_1}$, and for each $t \in [0, b]$ define f_t and g_t as

$$\begin{cases} f_t = \dfrac{t(b+m)-bm-\sqrt{bm(b-t)(m-t)}}{t} \\ g_t = \dfrac{t(b-m)+\sqrt{bt(b-m)(t-m)}}{b-t}. \end{cases}$$

It is easy to show that under the conditions of Lemma 2.3, it is true that $\nu_1 \geqslant m$, $o' \geqslant m$ and $\nu_1 \geqslant b'$.

For each $t \in [0, b]$, consider the equation $x^3 + A_t x^2 + B_t x + C_t = 0$ with

$$A_t = -\frac{1}{2}(2\nu_1 + m + 3t),$$
$$B_t = (2\nu_1 + m)t, \quad \text{and}$$
$$C_t = \frac{1}{2}(m\nu_2 - t\nu_2 - 2mt\nu_1).$$

Define x_t as the unique root of this equation in the interval $[0, \min(b', t)]$ if $t \leqslant m$, and y_t as the unique root of this equation in the interval $[\max(o', t), b]$ if $t > m$.

Theorem 3.2. *Consider a unimodal random variable X with support* $[0, b]$*, for which the first two moments are given by* μ_1 *and* μ_2 *and the mode by m.*

If $b' > m$*, then the tail probability is bounded as*

$$f_{low}^{(2)}(t) \leqslant \text{Prob}(X \geqslant t) \leqslant f_{upp}^{(2)}(t)$$

with $f_{low}^{(2)}$ *and* $f_{upp}^{(2)}$ *as in Table 2.*

If $b' \leqslant m$*, then the tail probability is bounded as*

$$f_{low}^{(3)}(t) \leqslant \text{Prob}(X \geqslant t) \leqslant f_{upp}^{(3)}(t)$$

with $f_{low}^{(3)}$ *and* $f_{upp}^{(3)}$ *as in Table 3.*

4. General restrictions on the Value at Risk

The results for the bounds for the Value at Risk are brought together in the next section. The calculations of the inversions as mentioned in Lemmas 2.4 and 2.5 are straightforward, although sometimes lengthy. Therefore, complete proofs are not provided in this paper. An example of how the inversions can be carried out is given in Appendix C. In the present paper, we only include full results where three parameters are known. For the cases up to two parameters, where the bounds are obviously less accurate, the results can be found in a technical report [18]. In Section 5, where we present numerical and graphical illustrations, bounds will be presented for all situations, including those where fewer than three parameters are known.

Table 3
Upper and lower bounds for $\mathrm{Prob}(X \geqslant t)$ (Theorem 3.2 — second case).

t	$f_{low}^{(3)}(t)$
$0 < t \leqslant \frac{m\nu_2}{\nu_2+2m\nu_1}$	$\frac{\nu_1^2 t+\nu_2(m-t)}{m\nu_2}$
$\frac{m\nu_2}{\nu_2+2m\nu_1} < t \leqslant \frac{mb+mb'-2b'^2}{b+2m-3b'}$	$\frac{m-t}{m-x_t} \cdot \frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1 x_t+x_t^2} + \frac{(\nu_1-x_t)^2}{\nu_2-2\nu_1 x_t+x_t^2}$
$\frac{mb+mb'-2b'^2}{b+2m-3b'} < t \leqslant \frac{bm(b+m-2b')}{b(b+m-2b')+(m-b')^2}$	$\frac{\nu_1-b'}{b-b'} + \frac{(b-\nu_1)(m-t)}{(b-b')(m-b')}$
$\frac{bm(b+m-2b')}{b(b+m-2b')+(m-b')^2} < t \leqslant m$	$\frac{m-t}{m} \cdot \frac{\nu_2-\nu_1(b+f_t)+bf_t}{bf_t} + \frac{m-t}{m-f_t} \cdot \frac{b\nu_1-\nu_2}{f_t(b-f_t)} + \frac{\nu_2-\nu_1 f_t}{b(b-f_t)}$
$m < t \leqslant \frac{\nu_2}{\nu_1}$	$\frac{\nu_2-t\nu_1}{b(b-m)}$
$\frac{\nu_2}{\nu_1} < t \leqslant b$	0
t	$f_{upp}^{(3)}(t)$
$0 < t \leqslant b'$	1
$b' < t \leqslant m$	$\frac{b(m-t)+(b+t)\nu_1-\nu_2}{mb}$
$m < t \leqslant \frac{bo'^2}{b(2o'-m)+(o'-m)^2}$	$\frac{(b\nu_1-\nu_2)(g_t-t)}{g_t(b-g_t)(g_t-m)} + \frac{(\nu_2-\nu_1 g_t)(b-t)}{b(b-g_t)(b-m)}$
$\frac{bo'^2}{b(2o'-m)+(o'-m)^2} < t \leqslant \frac{o'(2o'-m)}{3o'-2m}$	$\frac{\nu_1^2(\nu_2-t\nu_1)}{\nu_2(\nu_2-m\nu_1)}$
$\frac{o'(2o'-m)}{3o'-2m} < t \leqslant \frac{b(2b-m)-b'm}{3b-2m-b'}$	$\frac{y_t-t}{y_t-m} \cdot \frac{\nu_2-\nu_1^2}{y_t^2-2\nu_1 y_t+\nu_2}$
$\frac{b(2b-m)-b'm}{3b-2m-b'} < t \leqslant b$	$\frac{b-t}{b-m} \cdot \frac{\nu_2-\nu_1^2}{b^2-2\nu_1 b+\nu_2}$

4.1. *Knowledge of the first three moments*

Define κ_1, κ_2 and s_t as in 3.1.
For notational reasons define

- $\alpha_p = -\mu_1\left((1-p)\mu_2 - \mu_1^2\right)$
- $\beta_p = (1-p)(\mu_1\mu_3 + 2\mu_2^2) - 3\mu_1^2\mu_2$
- $\gamma_p = -3\mu_2\left((1-p)\mu_3 - \mu_1\mu_2\right)$
- $\delta_p = (1-p)\mu_3^2 - \mu_2^3$
- $\tilde{\alpha}_p = (1-p)\mu_2 - \mu_1^2$
- $\tilde{\beta}_p = \mu_1\mu_2 - (1-p)\mu_3$
- $\tilde{\gamma}_p = \mu_1\mu_3 - \mu_2^2$.

Theorem 4.1. *Consider a random variable X with support $[0, b]$, for which the first three moments are given by μ_1, μ_2 and μ_3. Then the Value at Risk is bounded as*

$$g_{low}^{(1)}(p) \leqslant \mathrm{VaR}_p(X) \leqslant g_{upp}^{(1)}(p)$$

with $g_{low}^{(1)}$ and $g_{upp}^{(1)}$ as in Table 4.

Theorem 4.2. *Consider a random variable X with support $\mathbb{R}^+$, for which the first three moments are given by μ_1, μ_2 and μ_3. Then the Value at Risk is bounded as*

$$g_{low}^{(2)}(p) \leqslant \mathrm{VaR}_p(X) \leqslant g_{upp}^{(2)}(p)$$

with $g_{low}^{(2)}$ and $g_{upp}^{(2)}$ as in Table 5.

4.2. *Knowledge of the first two moments and of the mode*

Define $f_t, g_t, \nu_1, \nu_2, x_t$ and y_t as in 3.2.
In order not to complicate the formulae, we will use a short notation for the following intervals:

- $I_1 = \left(\frac{m\nu_2}{\nu_2+2m\nu_1}, m\right]$
- $I_2 = \left(\frac{bb'^2}{b(2b'-m)+(b'-m)^2}, \frac{bo'^2}{b(2o'-m)+(o'-m)^2}\right]$
- $I_3 = \left(\frac{o'(2o'-m)}{3o'-2m}, \frac{b(2b-m)-b'm}{3b-2m-b'}\right]$
- $I_4 = \left(\frac{m\nu_2}{\nu_2+2m\nu_1}, \frac{mb+mb'-2b'^2}{b+2m-3b'}\right]$

Table 4
Upper and lower bounds for $\mathrm{VaR}_p(X)$ (Theorem 4.1 — finite support).

p	$g_{low}^{(1)}(p)$
$0 < p \leqslant 1 - \frac{\mu_1\mu_3-\mu_2^2}{b(b^2\mu_1-2b\mu_2+\mu_3)} - \frac{(b\mu_1-\mu_2)^3}{(b\mu_2-\mu_3)(b^2\mu_1-2b\mu_2+\mu_3)}$	0
$1 - \frac{\mu_1\mu_3-\mu_2^2}{b(b^2\mu_1-2b\mu_2+\mu_3)} - \frac{(b\mu_1-\mu_2)^3}{(b\mu_2-\mu_3)(b^2\mu_1-2b\mu_2+\mu_3)} < p \leqslant 1 - \frac{(\mu_2-\kappa_1\mu_1)^3}{(\mu_3-\kappa_1\mu_2)(\mu_3-2\kappa_1\mu_2+\kappa_1^2\mu_1)}$	Unique solution in $(0, c_1]$ for t implicitly from $1 - p = \frac{\mu_2-\mu_1(t+b)+tb}{(t-s_t)(b-s_t)} + \frac{\mu_2-\mu_1(s_t+t)+s_t t}{(b-s_t)(b-t)}$
$1 - \frac{(\mu_2-\kappa_1\mu_1)^3}{(\mu_3-\kappa_1\mu_2)(\mu_3-2\kappa_1\mu_2+\kappa_1^2\mu_1)} < p \leqslant 1 - \frac{\mu_1\mu_3-\mu_2^2}{b(b^2\mu_1-2b\mu_2+\mu_3)}$	Unique root in $\left(\kappa_1, \frac{b\mu_2-\mu_3}{b\mu_1-\mu_2}\right]$ of the equation $\alpha_p t^3 + \beta_p t^2 + \gamma_p t + \delta_p = 0$
$1 - \frac{\mu_1\mu_3-\mu_2^2}{b(b^2\mu_1-2b\mu_2+\mu_3)} < p < 1$	Unique solution in $\left(\frac{b\mu_2-\mu_3}{b\mu_1-\mu_2}, \kappa_2\right)$ for t implicitly from $1 - p = \frac{\mu_2-\mu_1(s_t+t)+s_t t}{(b-s_t)(b-t)}$
p	$g_{upp}^{(1)}(p)$
$0 < p \leqslant 1 - \frac{\mu_1(b\mu_1-\mu_2)}{b\mu_2-\mu_3} + \frac{\mu_1\mu_3-\mu_2^2}{b(b\mu_2-\mu_3)}$	Unique root in $\left(\kappa_1, \frac{b\mu_2-\mu_3}{b\mu_1-\mu_2}\right]$ of the equation $\tilde{\alpha}_p t^2 + \tilde{\beta}_p t + \tilde{\gamma}_p = 0$
$1 - \frac{\mu_1(b\mu_1-\mu_2)}{b\mu_2-\mu_3} + \frac{\mu_1\mu_3-\mu_2^2}{b(b\mu_2-\mu_3)} < p \leqslant 1 - \frac{\mu_3\mu_1-\mu_2^2}{\kappa_2(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1)}$	Unique solution in $\left(\frac{b\mu_2-\mu_3}{b\mu_1-\mu_2}, \kappa_2\right]$ for t implicitly from $1 - p = \frac{\mu_2-\mu_1(s_t+t)+s_t t}{(b-s_t)(b-t)} + \frac{\mu_2-\mu_1(s_t+b)+s_t b}{(s_t-t)(b-t)}$
$1 - \frac{\mu_3\mu_1-\mu_2^2}{\kappa_2(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1)} < p \leqslant 1 - \frac{\mu_3\mu_1-\mu_2^2}{b(\mu_3-2b\mu_2+b^2\mu_1)}$	Unique root in$(\kappa_2, b]$ of the equation $\mu_1 t^3 - 2\mu_2 t^2 + \mu_3 t - \frac{\mu_3\mu_1-\mu_2^2}{1-p} = 0$
$1 - \frac{\mu_3\mu_1-\mu_2^2}{b(\mu_3-2b\mu_2+b^2\mu_1)} < p < 1$	b

Table 5
Upper and lower bounds for $\mathrm{VaR}_p(X)$ (Theorem 4.2 — infinite support).

p	$g_{low}^{(2)}(p)$
$0 < p \leqslant \frac{\mu_2-\mu_1^2}{\mu_2}$	0
$\frac{\mu_2-\mu_1^2}{\mu_2} < p \leqslant 1 - \frac{(\mu_2-\kappa_1\mu_1)^3}{(\mu_3-\kappa_1\mu_2)(\mu_3-2\kappa_1\mu_2+\kappa_1^2\mu_1)}$	$\mu_1 - \sqrt{\frac{1-p}{p}(\mu_2-\mu_1^2)}$
$1 - \frac{(\mu_2-\kappa_1\mu_1)^3}{(\mu_3-\kappa_1\mu_2)(\mu_3-2\kappa_1\mu_2+\kappa_1^2\mu_1)} < p < 1$	Unique root in $\left(\kappa_1, \frac{\mu_2}{\mu_1}\right]$ of the equation $\alpha_p t^3 + \beta_p t^2 + \gamma_p t + \delta_p = 0$
p	$g_{upp}^{(2)}(p)$
$0 < p \leqslant \frac{\mu_2-\mu_1^2}{\mu_2}$	Unique root in $\left(\kappa_1, \frac{\mu_2}{\mu_1}\right]$ of the equation $\tilde{\alpha}_p t^2 + \tilde{\beta}_p t + \tilde{\gamma}_p = 0$
$\frac{\mu_2-\mu_1^2}{\mu_2} < p \leqslant 1 - \frac{\mu_3\mu_1-\mu_2^2}{\kappa_2(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1)}$	$\mu_1 + \sqrt{\frac{p}{1-p}(\mu_2-\mu_1^2)}$
$1 - \frac{\mu_3\mu_1-\mu_2^2}{\kappa_2(\mu_3-2\kappa_2\mu_2+\kappa_2^2\mu_1)} < p < 1$	Unique root in$(\kappa_2, +\infty)$ of the equation $\mu_1 t^3 - 2\mu_2 t^2 + \mu_3 t - \frac{\mu_3\mu_1-\mu_2^2}{1-p} = 0$

- $I_5 = \left(\frac{bm(b+m-2b')}{b(b+m-2b')+(m-b')^2}, m\right]$
- $I_6 = \left(m, \frac{bo'^2}{b(2o'-m)+(o'-m)^2}\right]$
- $I_7 = \left(\frac{o'(2o'-m)}{3o'-2m}, \frac{b(2b-m)-b'm}{3b-2m-b'}\right]$
- $I_8 = \left(\frac{v_2(2v_2-mv_1)}{v_1(3v_2-2mv_1)}, +\infty\right)$.

Theorem 4.3. *Consider a unimodal random variable X with support* $[0, b]$, *for which the first two moments are given by* μ_1 *and* μ_2 *and the mode by m.*

If $b' > m$, *then the Value at Risk is bounded as*

$$g_{low}^{(3)}(p) \leqslant \mathrm{VaR}_p(X) \leqslant g_{upp}^{(3)}(p)$$

with $g_{low}^{(3)}$ *and* $g_{upp}^{(3)}$ *as in Table 6.*

If $b' \leqslant m$, *then the Value at Risk is bounded as*

$$g_{low}^{(4)}(p) \leqslant \mathrm{VaR}_p(X) \leqslant g_{upp}^{(4)}(p)$$

with $g_{low}^{(4)}$ *and* $g_{upp}^{(4)}$ *as in Table 7.*

Table 6
Upper and lower bounds for $\mathrm{VaR}_p(X)$ (Theorem 4.3 — finite support).

p	$g_{low}^{(3)}(p)$
$0 < p \leqslant \frac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}$	$\frac{m\nu_2 p}{\nu_2-\nu_1^2}$
$\frac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1} < p \leqslant \frac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2}$	Unique solution in $\left(\frac{m\nu_2}{\nu_2+2m\nu_1}, m\right]$ for t implicitly from $1-p = \frac{m-t}{m-x_t} \cdot \frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1 x_t+x_t^2} + \frac{(\nu_1-x_t)^2}{\nu_2-2\nu_1 x_t+x_t^2}$
$\frac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2} < p \leqslant \frac{\nu_2-\nu_1^2+(b'-m)(\nu_1-b')}{\nu_2-\nu_1^2+(\nu_1-m)(\nu_1-b')}$	$\nu_1 - \frac{1}{2}(1-p)(\nu_1-m) - \frac{1}{2}\sqrt{(1-p)^2(\nu_1-m)^2+4(1-p)(\nu_2-\nu_1^2)}$
$\frac{\nu_2-\nu_1^2+(b'-m)(\nu_1-b')}{\nu_2-\nu_1^2+(\nu_1-m)(\nu_1-b')} < p < 1$	$\frac{\nu_2-(1-p)b(b-m)}{\nu_1}$
p	$g_{upp}^{(3)}(p)$
$0 < p \leqslant 1 - \frac{bb'+\nu_1(b'-m)}{b(2b'-m)+(b'-m)^2}$	$\frac{b(\nu_1-b')(b'-m)+b'(b-\nu_1)(b-m)}{(\nu_1-b')(b'-m)+(b-\nu_1)(b-m)} - \frac{(1-p)(b-b')(b-m)(b'-m)}{(\nu_1-b')(b'-m)+(b-\nu_1)(b-m)}$
$1 - \frac{bb'+\nu_1(b'-m)}{b(2b'-m)+(b'-m)^2} < p \leqslant 1 - \frac{\nu_1^2(\nu_2 b(2o'-m)+\nu_2(o'-m)^2-b\nu_1 o'^2)}{\nu_2(\nu_2-m\nu_1)(b(2o'-m)+(o'-m)^2)}$	Unique solution in I_1 for t implicitly from $1-p = \frac{(b\nu_1-\nu_2)(g_t-t)}{g_t(b-g_t)(g_t-m)} + \frac{(\nu_2-\nu_1 g_t)(b-t)}{b(b-g_t)(b-m)}$
$1 - \frac{\nu_1^2(\nu_2 b(2o'-m)+\nu_2(o'-m)^2-b\nu_1 o'^2)}{\nu_2(\nu_2-m\nu_1)(b(2o'-m)+(o'-m)^2)} < p \leqslant 1 - \frac{\nu_1^2(\nu_2(3o'-2m)-\nu_1 o'(2o'-m))}{\nu_2(\nu_2-m\nu_1)(3o'-2m)}$	$\frac{\nu_2}{\nu_1^3}\left(\nu_1^2-(1-p)(\nu_2-m\nu_1)\right)$
$1 - \frac{\nu_1^2(\nu_2(3o'-2m)-\nu_1 o'(2o'-m))}{\nu_2(\nu_2-m\nu_1)(3o'-2m)} < p \leqslant 1 - \frac{(\nu_2-\nu_1^2)(b-b')}{(3b-2m-b')(b^2-2\nu_1 b+\nu_2)}$	Unique solution in I_2 for t implicitly from $1-p = \frac{y_t-t}{y_t-m} \cdot \frac{\nu_2-\nu_1^2}{y_t^2-2\nu_1 y_t+\nu_2}$
$1 - \frac{(\nu_2-\nu_1^2)(b-b')}{(3b-2m-b')(b^2-2\nu_1 b+\nu_2)} < p < 1$	$b-(1-p)(b-m)\frac{b^2-2\nu_1 b+\nu_2}{\nu_2-\nu_1^2}$

Table 7
Upper and lower bounds for $\mathrm{VaR}_p(X)$ (Theorem 4.3 — finite support).

p	$g_{low}^{(4)}(p)$
$0 < p < \frac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}$	$\frac{m\nu_2 p}{\nu_2-\nu_1^2}$
$\frac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1} < p < \frac{b-\nu_1}{b+2m-3b'}$	Unique solution in $I_3$for t implicitly from $1-p = \frac{m-t}{m-x_t} \cdot \frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1 x_t+x_t^2} + \frac{(\nu_1-x_t)^2}{\nu_2-2\nu_1 x_t+x_t^2}$
$\frac{b-\nu_1}{b+2m-3b'} < p < \frac{b^2-\nu_1(m-b')-b(\nu_1-m+b')}{b^2+(m-b')^2+b(m-2b')}$	$m - \frac{m-b'}{b-\nu_1}\left((1-p)(b-b')-(\nu_1-b')\right)$
$\frac{b^2-\nu_1(m-b')-b(\nu_1-m+b')}{b^2+(m-b')^2+b(m-2b')} < p < 1 - \frac{\nu_2-m\nu_1}{b(b-m)}$	Unique solution in $I_4$for t implicitly from $1-p = \frac{m-t}{m} \cdot \frac{\nu_2-\nu_1(b+f_t)+bf_t}{bf_t} + \frac{m-t}{m-f_t} \cdot \frac{b\nu_1-\nu_2}{f_t(b-f_t)} + \frac{\nu_2-\nu_1 f_t}{b(b-f_t)}$
$1 - \frac{\nu_2-m\nu_1}{b(b-m)} < p < 1$	$\frac{\nu_2-(1-p)b(b-m)}{\nu_1}$
p	$g_{upp}^{(4)}(p)$
$0 < p \leqslant 1 - \frac{(b+m)\nu_1-\nu_2}{mb}$	$\frac{b(m+\nu_1)-\nu_2-(1-p)mb}{b-\nu_1}$
$1 - \frac{(b+m)\nu_1-\nu_2}{mb} < p \leqslant 1 - \frac{\nu_1^2(\nu_2 b(2o'-m)+\nu_2(o'-m)^2-b\nu_1 o'^2)}{\nu_2(\nu_2-m\nu_1)(b(2o'-m)+(o'-m)^2)}$	Unique solution in I_5 for t implicitly from $1-p = \frac{(b\nu_1-\nu_2)(g_t-t)}{g_t(b-g_t)(g_t-m)} + \frac{(\nu_2-\nu_1 g_t)(b-t)}{b(b-g_t)(b-m)}$
$1 - \frac{\nu_1^2(\nu_2 b(2o'-m)+\nu_2(o'-m)^2-b\nu_1 o'^2)}{\nu_2(\nu_2-m\nu_1)(b(2o'-m)+(o'-m)^2)} < p \leqslant 1 - \frac{\nu_1^2(\nu_2(3o'-2m)-\nu_1 o'(2o'-m))}{\nu_2(\nu_2-m\nu_1)(3o'-2m)}$	$\frac{\nu_2}{\nu_1^3}\left(\nu_1^2-(1-p)(\nu_2-m\nu_1)\right)$
$1 - \frac{\nu_1^2(\nu_2(3o'-2m)-\nu_1 o'(2o'-m))}{\nu_2(\nu_2-m\nu_1)(3o'-2m)} < p \leqslant 1 - \frac{(\nu_2-\nu_1^2)(b-b')}{(3b-2m-b')(b^2-2\nu_1 b+\nu_2)}$	Unique solution in I_6 for t implicitly from $1-p = \frac{y_t-t}{y_t-m} \cdot \frac{\nu_2-\nu_1^2}{y_t^2-2\nu_1 y_t+\nu_2}$
$1 - \frac{(\nu_2-\nu_1^2)(b-b')}{(3b-2m-b')(b^2-2\nu_1 b+\nu_2)} < p < 1$	$b-(1-p)(b-m)\frac{b^2-2\nu_1 b+\nu_2}{\nu_2-\nu_1^2}$

Theorem 4.4. *Consider a unimodal random variable X with support $\mathbb{R}^+$, for which the first two moments are given by μ_1 and μ_2 and the mode by m. Then the Value at Risk is bounded as*

$$g_{low}^{(5)}(p) \leqslant \mathrm{VaR}_p(X) \leqslant g_{upp}^{(5)}(p)$$

with $g_{low}^{(5)}$ and $g_{upp}^{(5)}$ as in Table 8.

Note that in case the support of the variable X coincides with the whole positive real line, it is always true that $b' = \nu_1 \geqslant m$.

5. Numerical illustrations

5.1. Example in the case of a variable with finite support

Suppose a unimodal variable X has support $[0, 200]$, with parameters $m = 7$, $\mu_1 = 10$, $\mu_2 = 240$ and $\mu_3 = 14000$, which corresponds to a standard deviation equal to $\sigma = 11.8322$ and a skewness $\gamma_1 = 5.3124$.

Table 8
Upper and lower bounds for $\mathrm{VaR}_p(X)$ (Theorem 4.4 — infinite support).

p	$g_{low}^{(5)}(p)$
$0 < p \leqslant \frac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1}$	$\frac{m\nu_2 p}{\nu_2-\nu_1^2}$
$\frac{\nu_2-\nu_1^2}{\nu_2+2m\nu_1} < p \leqslant \frac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2}$	Unique solution in $\left(\frac{m\nu_2}{\nu_2+2m\nu_1}, m\right]$ for t implicitly from $1-p=\frac{m-t}{m-x_t}\cdot\frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1x_t+x_t^2}+\frac{(\nu_1-x_t)^2}{\nu_2-2\nu_1x_t+x_t^2}$
$\frac{\nu_2-\nu_1^2}{\nu_2-\nu_1^2+(\nu_1-m)^2} < p < 1$	$\nu_1-\frac{1}{2}(1-p)(\nu_1-m)-\frac{1}{2}\sqrt{(1-p)^2(\nu_1-m)^2+4(1-p)(\nu_2-\nu_1^2)}$

p	$g_{upp}^{(5)}(p)$
$0 < p \leqslant \frac{\nu_1-m}{2\nu_1-m}$	$p\nu_1+(1-p)m$
$\frac{\nu_1-m}{2\nu_1-m} < p \leqslant 1-\frac{\nu_1^2}{2\nu_2-m\nu_1}$	$\frac{\nu_1^2+(1-p)^2m^2+2m\nu_1(1-p)}{4\nu_1(1-p)}$
$1-\frac{\nu_1^2}{2\nu_2-m\nu_1} < p \leqslant 1-\frac{\nu_1^2}{3\nu_2-2m\nu_1}$	$\frac{\nu_2}{\nu_1^3}\left(\nu_1^2-(1-p)(\nu_2-m\nu_1)\right)$
$1-\frac{\nu_1^2}{3\nu_2-2m\nu_1} < p < 1$	Unique solution in I_7 for t implicitly from $1-p=\frac{y_t-t}{y_t-m}\cdot\frac{\nu_2-\nu_1^2}{y_t^2-2\nu_1y_t+\nu_2}$

Table 9
Bounds for the Value at Risk for high percentiles in the finite case.

p (%)		μ_1	m	μ_1, μ_2	μ_1, m	μ_1, μ_2, μ_3	μ_1, μ_2, m
90.0	L.B.	0.000	6.300	6.056	6.738	6.364	**6.996**
	U.B.	100.000	180.700	45.497	36.094	41.389	**31.773**
	Width	100.000	174.400	39.441	29.356	35.025	**24.777**
92.5	L.B.	0.000	6.475	6.631	6.896	6.834	**7.128**
	U.B.	133.333	185.525	51.553	46.904	47.604	**36.134**
	Width	133.333	179.050	44.922	40.008	40.770	**29.006**
95.0	L.B.	0.000	6.650	7.286	6.981	7.375	**8.314**
	U.B.	200.000	190.350	61.575	68.547	58.587	**43.186**
	Width	200.000	183.700	54.289	61.566	51.212	**34.872**
97.5	L.B.	5.128	6.825	8.105	8.175	9.538	**9.719**
	U.B.	200.000	195.175	83.892	125.569	80.977	**58.465**
	Width	194.872	188.350	75.787	117.394	71.439	**48.746**
99.0	L.B.	8.081	6.930	8.811	11.070	14.066	**14.923**
	U.B.	200.000	198.070	127.729	170.308	106.949	**87.859**
	Width	191.919	191.140	118.918	159.238	92.883	**72.936**

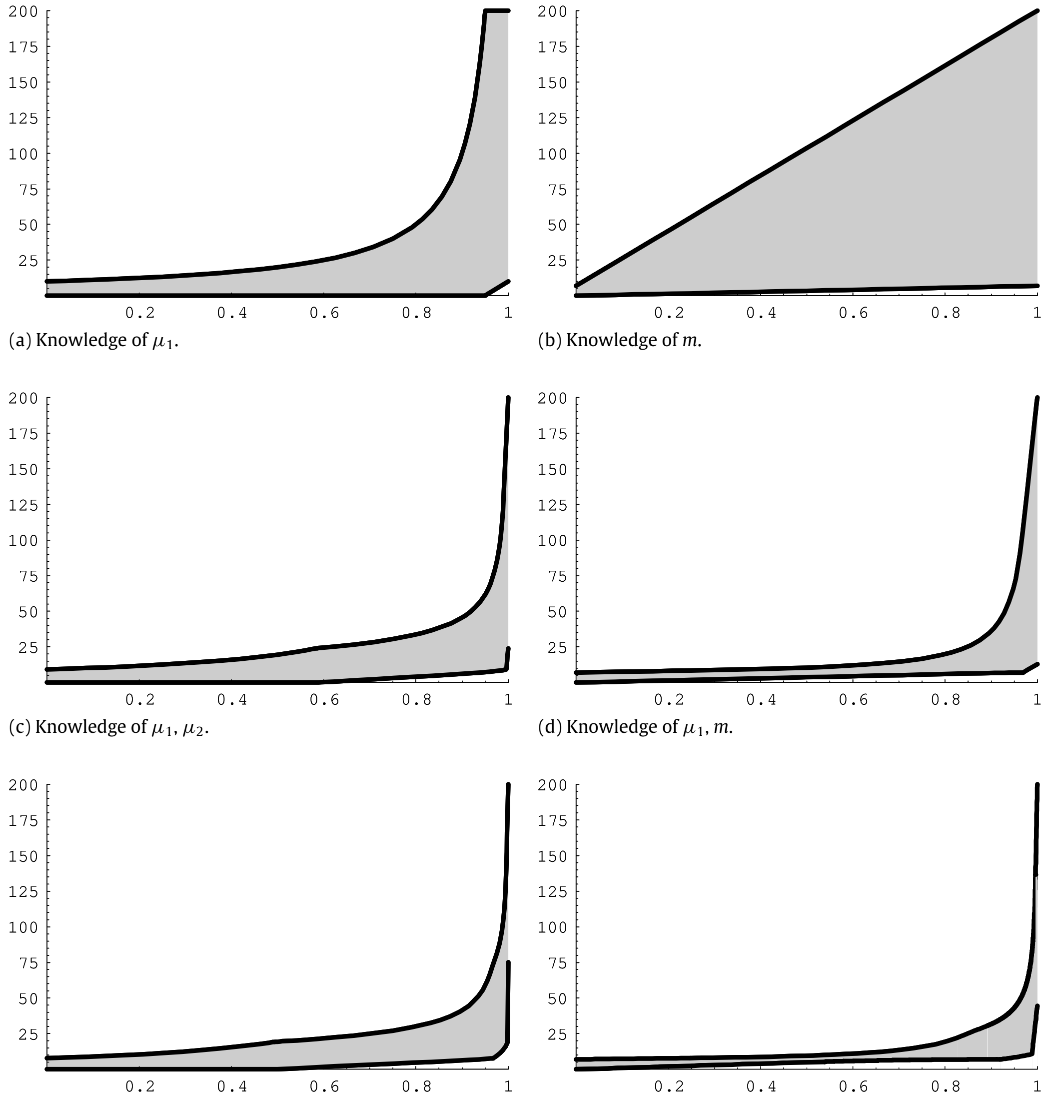
Applying the methodology of the previous sections, we can establish general restrictions on the possible outcomes of the Value at Risk. We present the results in two ways. In Fig. 1 we show graphs of the general restrictions on the Value at Risk where one or more of the parameters of the distribution are fixed, for the whole range of the variable X. In Table 9 we show the explicit results for the minimum and maximum value, as well as the width of the interval estimate for the Value at Risk, for some common percentiles between 90% en 99%. For each choice of the percentiles, the sharpest bounds are indicated in bold.

On the basis of the numerical results in Table 9 and Fig. 1, we can formulate several conclusions. Evidently, the lower and upper bounds increase with the percentile p. Except for very high percentiles, the bounds become quite accurate in the last plots, even though, besides the range of the variable, we only fix a maximum of three distribution parameters. It is also obvious that the introduction of an extra parameter improves the lower and upper bound for the Value at Risk. However, it is important to note that it is not true, for example, that the feasible range in the case of knowledge of two moments and mode (Figure (f)) would be just the intersection of the range for two given moments (Figure (c)) and the range for a given mode (Figure (b)). Note also that in the case of two or three known parameters, the influence of the mode is much more important than the influence of an extra moment. Indeed, the ranges shown in Figures (d) and (f) are significantly more accurate than those of Figures (c) and (e).

5.2. Example in the case of a variable with infinite support

We now consider a unimodal variable X with support $\mathbb{R}^+$, and with parameters $m = 2.6896$, $\mu_1 = 10$, $\mu_2 = 240$ and $\mu_3 = 13824$, reflecting a standard deviation $\sigma = 11.8322$ and a skewness $\gamma_1 = 5.2062$. These parameters are chosen in such a way that they correspond to the moments and mode of a lognormal variable, as this type of distribution is often used for modeling purposes.

Applying the methodology of the previous sections, we can again establish general restrictions on the possible outcomes of the Value at Risk. In Table 10 we show the explicit results for the minimum and maximum value, as well as the width



(a) Knowledge of μ_1.

(b) Knowledge of m.

(c) Knowledge of μ_1, μ_2.

(d) Knowledge of μ_1, m.

(e) Knowledge of μ_1, μ_2, μ_3.

(f) Knowledge of μ_1, μ_2, m.

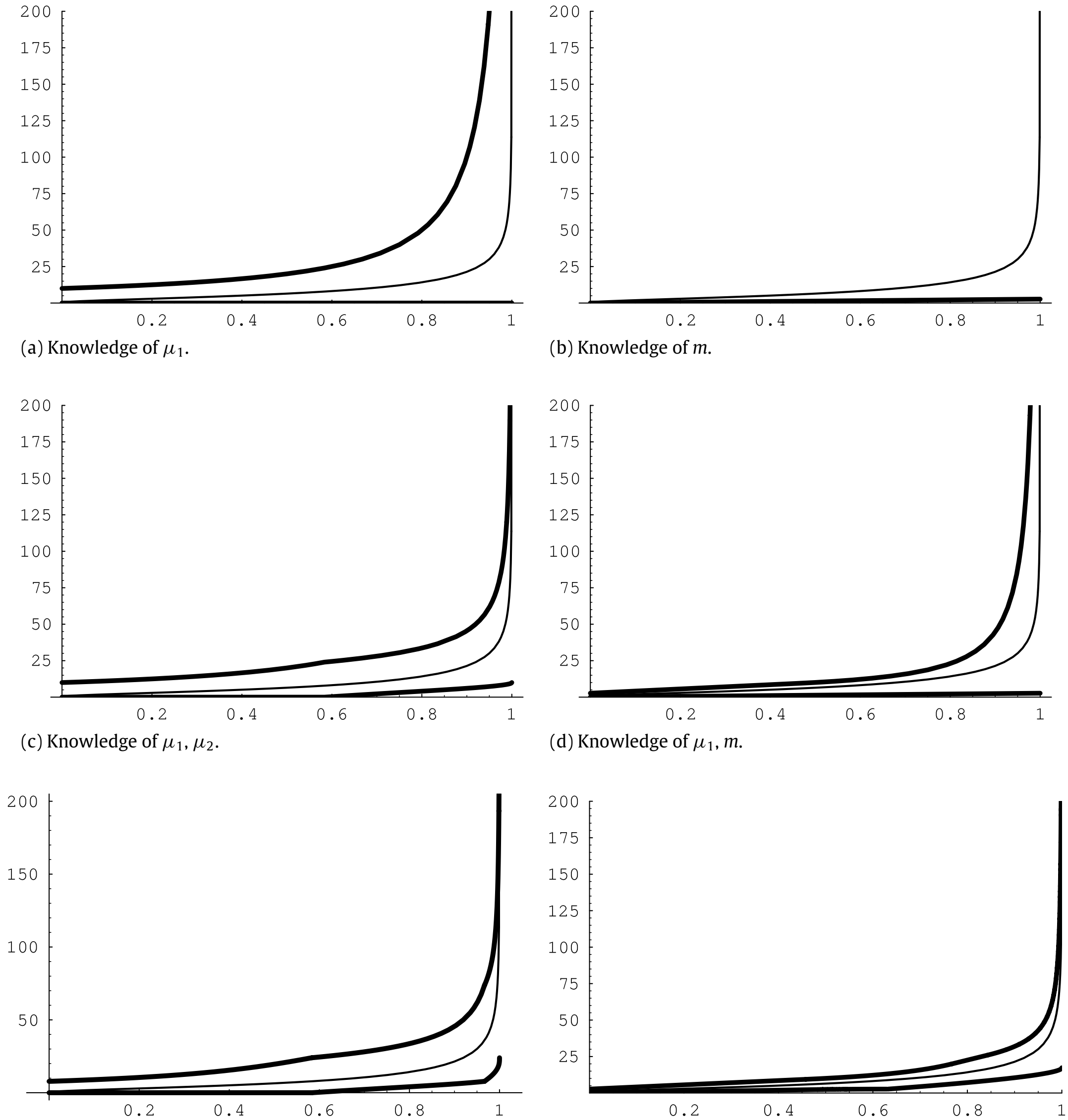
Fig. 1. Restrictions on the Value at Risk in the finite case if one, two or three parameters are fixed.

of the interval estimate for the Value at Risk, for the same percentiles as in the previous example. For each choice of the percentiles, the sharpest bounds are indicated in bold. In the last column, the exact value for the Value at Risk is shown in the case of a lognormal distribution which fits the specific parameter values. In Fig. 2, we illustrate the results by means of a graphical representation. In each diagram, the upper and lower bounds for the Value at Risk are depicted, together with the quantiles of the corresponding lognormal distribution.

We can repeat the remarks as for the example in the finite case. The ranges between upper and lower bound are rather tight, except for very high percentiles. However, we see that if we compare the bounds with the quantiles of a lognormal distribution, these quantiles are much closer to the upper bound than to the lower bound. If we take into account the fact that the tails of a lognormal distribution are not always as heavy as they should be for real applications, this suggests that even with a somewhat wider range, the upper bounds can still be used as a rather reliable measure for the true Value at Risk.

6. Conclusion

In the present contribution, we derived general restrictions for the Value at Risk of a random variable, by fixing only some specific parameters (moments and/or mode) of the underlying probability distribution. Starting with bounds on tail probabilities, we showed how these bounds can be converted into bounds for the Value at Risk. These bounds apply for all possible underlying distributions with the specified successive moments and/or mode. Detailed analytical results are given



(a) Knowledge of μ_1.

(b) Knowledge of m.

(c) Knowledge of μ_1, μ_2.

(d) Knowledge of μ_1, m.

(e) Knowledge of μ_1, μ_2, μ_3.

(f) Knowledge of μ_1, μ_2, m.

Fig. 2. Restrictions on the Value at Risk in the infinite case if one, two or three parameters are fixed.

for cases where three parameters are known; for the numerical illustrations, these results are compared to situations where less than three parameters are fixed. For variables with an infinite support, the numerical results are also compared with the outcomes of a lognormal model, as this model is rather common in practical applications.

From the numerical illustrations, it can be seen that, with a maximum of three given parameters, the best estimates are achieved with a unimodal random variable for which two moments and the mode or given. For random variables with an infinite support, upper and lower bounds for the Value at Risk are more diverging for very high percentiles. However, by comparing the bounds with the estimates for the Value at Risk in the case of the lognormal model, which is commonly used in practical applications, it turns out that in that case the upper bound is clearly more accurate than the lower bound. Taking into account the fact that in many situations the lognormal distribution is less fat-tailed than it should be, this proves that our upper bounds can provide useful and valuable information.

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Table 10

p (%)		μ_1	m	μ_1, μ_2	μ_1, m	μ_1, μ_2, μ_3	μ_1, μ_2, m	Lognormal
90	L.B.	0.000	2.421	6.056	2.421	6.056	**10.481**	*21.412*
	U.B.	100.000	∞	45.497	44.631	45.497	**31.944**	*21.412*
	Width	100.000	∞	39.441	42.210	39.441	**21.463**	
92.5	L.B.	0.000	2.488	6.631	2.488	6.631	**11.490**	*24.823*
	U.B.	133.333	∞	51.553	59.054	51.553	**36.165**	*24.823*
	Width	133.333	∞	44.922	56.566	44.922	**24.675**	
95	L.B.	0.000	2.555	7.286	2.555	7.286	**12.648**	*30.081*
	U.B.	200.000	∞	61.575	87.902	61.575	**42.903**	*30.081*
	Width	200.000	∞	54.289	85.347	54.289	**30.255**	
97.5	L.B.	0.000	2.622	8.105	2.622	9.740	**14.095**	*40.396*
	U.B.	400.000	∞	83.892	174.452	80.551	**57.383**	*40.396*
	Width	400.000	∞	75.787	171.830	70.811	**43.288**	
99	L.B.	0.000	2.663	8.811	2.663	14.205	**15.321**	*56.914*
	U.B.	1000.00	∞	127.729	434.107	106.327	**85.135**	*56.914*
	Width	1000.00	∞	118.918	431.444	92.122	**69.814**	

Appendix A. Construction of two and three point distributions

In Section 2.1 we indicated that one crucial aspect of the calculation of upper and lower bounds is the construction of two and three point distributions, as it is for this type of distributions that the bounds can be reached.

If we look for point distributions for which the first two moments are fixed, the following results can be used (for a proof, see [14]):

Lemma A.1. *Consider μ_1 and μ_2, satisfying the conditions of Lemma 2.3 such that they can be considered as moments, and define $o' = \frac{\mu_2}{\mu_1}$ and $b' = \frac{\mu_1 b - \mu_2}{b-\mu_1}$.*

If $0 < r < b'$, and if $r' = \frac{\mu_1 r - \mu_2}{r-\mu_1}$, then a two point distribution exists with moments μ_1 and μ_2 on $[0, b]$ in r and r' with masses

$$q_r = \frac{\mu_1 - r'}{r - r'} \quad \text{and} \quad q_r' = \frac{\mu_1 - r}{r' - r}.$$

It is also true that $0 < r < b' < \mu_1 < o' < r' < b$.

If $b' < s < o'$, then a three point distribution exists with moments μ_1 and μ_2 on $[0, b]$ in 0, s and b with masses

$$q_s = \frac{b\mu_1 - \mu_2}{s(b-s)} \quad \text{and} \quad q_b = \frac{\mu_2 - \mu_1 s}{b(b-s)} \quad \text{and} \quad q_0 = 1 - q_s - q_b.$$

Lemma A.2. *Consider μ_1, μ_2 and μ_3, satisfying the conditions of Lemma 2.3 such that they can be considered as moments.*

If κ_1 and κ_2 are defined as in Section 3.1, then a two point distribution exists with moments μ_1, μ_2 and μ_3 on $[0, b]$ in κ_1 and κ_2 with masses

$$q_{\kappa_1} = \frac{\mu_1 - \kappa_2}{\kappa_1 - \kappa_2} \quad \text{and} \quad q_{\kappa_2} = \frac{\mu_1 - \kappa_1}{\kappa_2 - \kappa_1}.$$

If $0 < r < \kappa_1$ and $\kappa_2 < s < b$, and if $u = u(r, s)$ with

$$u(r, s) = \frac{\mu_3 - (r+s)\mu_2 + rs\mu_1}{\mu_2 - (r+s)\mu_1 + rs}, \tag{6}$$

then a three point distribution exists with moments μ_1, μ_2 and μ_3 on $[0, b]$ in r, u and s with masses

$$q_r = \frac{\mu_2 - (u+s)\mu_1 + us}{(r-u)(r-s)} \quad \text{and} \quad q_s = \frac{\mu_2 - (r+u)\mu_1 + ru}{(s-r)(s-u)} \quad \text{and} \quad q_u = \frac{\mu_2 - (r+s)\mu_1 + rs}{(u-r)(u-s)}.$$

It is also true that $0 < r < \kappa_1 < u < \kappa_2 < s < b$.

Appendix B. Proof of the results of Theorem 3.1

Theorem 3.1 is a direct application of the method described in Section 2.1. In what follows we call a polynomial P "appropriate" if it equals the function $1_{[t,b]}$ on $[0, b]$ in the mass points of the distribution and if it remains at one side (below or above) of $1_{[t,b]}$ on $[0, b]$.

In order to get the explicit results of the theorem, we can use the three point distribution as indicated in Lemma A.2. The boundary value $(b\mu_2 - \mu_3)/(b\mu_1 - \mu_2)$, which is one of the entries of t in Table 1, equals $u(0, b)$.

In the results of Theorem 3.1 the following cases can be distinguished.

(a) If $0 \leqslant t \leqslant \kappa_1 \leqslant \kappa_2$, obviously the best upper bound is equal to 1. A three point distribution also exists with masses in t, $u(t, b)$ and b and an appropriate polynomial of degree 3 remaining below $1_{[t,b]}$ on $[0, b]$, leading to $q_u + q_b$ as best lower bound.

(b) If $\kappa_1 \leqslant t \leqslant u(0, b)$, then a unique solution s of the equation $u(0, s) = t$ exists, generating a three point distribution with masses in 0, t and s. Two appropriate polynomials of degree 3 can be found, one remaining above and one remaining below $1_{[t,b]}$ on $[0, b]$, leading to q_s as best lower bound and $q_t + q_s$ as best upper bound.

(c) If $u(0, b) \leqslant t \leqslant \kappa_2$, then a unique solution r of the equation $u(r, b) = t$ exists, generating a three point distribution with masses in r, t and b. Two appropriate polynomials of degree 3 can be found, one remaining above and one remaining below $1_{[t,b]}$ on $[0, b]$, leading to q_b as best lower bound and $q_t + q_b$ as best upper bound.

(d) Finally if $\kappa_1 \leqslant \kappa_2 \leqslant t$, obviously the best lower bound equals 0. A three point distribution also exists with masses in 0, $u(0, t)$ and t and an appropriate polynomial of degree 3 one remaining above $1_{[t,b]}$ on $[0, b]$, leading to q_t as best upper bound.

After determining q_s, q_t, q_r and q_b according to Lemma A.2, the bounds on $\text{Prob}(X \geqslant t)$ as given in Theorem 3.1 and Table 1 can be obtained in a straightforward way.

Appendix C. Example of the calculation of the inversions as described in Section 2.3; Partial proof of the results of Theorems 4.3 and 4.4

In this section, we show how the results of the first part of Table 2 can be converted into the results of the first part of Tables 6 and 8. The other conversions can be proved in an analogous way.

Table 2 contains the results for the lower bounds for the tail probability in case two moments and the mode are fixed. We start with the calculation of the lower bounds in the switch points:

$$\begin{aligned}
t_0 &= 0 && \longmapsto && f_{low}^{(2)}(t_0) = 1 \\
t_1 &= \frac{m\nu_2}{\nu_2 + 2m\nu_1} && \longmapsto && f_{low}^{(2)}(t_1) = \frac{\nu_1(\nu_1 + 2m)}{\nu_2 + 2m\nu_1} \\
t_2 &= m && \longmapsto && f_{low}^{(2)}(t_2) = \frac{(\nu_1 - m)^2}{\nu_2 - \nu_1^2 + (\nu_1 - m)^2} \\
t_3 &= b' && \longmapsto && f_{low}^{(2)}(t_3) = \frac{(\nu_1 - b')^2}{\nu_2 - \nu_1^2 + (\nu_1 - m)(\nu_1 - b')} \\
t_4 &= \frac{\nu_2}{\nu_1} && \longmapsto && f_{low}^{(2)}(t_4) = 0.
\end{aligned}$$

These values can be transformed into switch points for the lower bounds for the Value at Risk as explained in Lemma 2.4:

$$\begin{aligned}
p_0 &= 1 - f_{low}^{(2)}(t_0) = 0 \\
p_1 &= 1 - f_{low}^{(2)}(t_1) = \frac{\nu_2 - \nu_1^2}{\nu_2 + 2m\nu_1} \\
p_2 &= 1 - f_{low}^{(2)}(t_2) = \frac{\nu_2 - \nu_1^2}{\nu_2 - \nu_1^2 + (\nu_1 - m)^2} \\
p_3 &= 1 - f_{low}^{(2)}(t_3) = \frac{\nu_2 - \nu_1^2 + (b' - m)(\nu_1 - b')}{\nu_2 - \nu_1^2 + (\nu_1 - m)(\nu_1 - b')} \\
p_4 &= 1 - f_{low}^{(2)}(t_4) = 1.
\end{aligned}$$

Adopting the approach of Lemma 2.4, the lower bounds for $\text{VaR}_p(X)$ as presented in Table 6 can be deduced in the following way:

- for $0 < p < p_1$:

$$f_{low}^{(2)}(t) = 1 - p \;\Leftrightarrow\; \frac{\nu_1^2 t + \nu_2(m - t)}{m\nu_2} = 1 - p$$

  solving for t results in $t = \frac{m\nu_2 p}{\nu_2 - \nu_1^2}$, which is the lower bound for $\text{VaR}_p(X)$;

- for $p_1 < p < p_2$:

$$f_{low}^{(2)}(t) = 1 - p \;\Leftrightarrow\; \frac{m-t}{m-x_t}\frac{\nu_2-\nu_1^2}{\nu_2-2\nu_1 x_t+x_t^2} + \frac{(\nu_1-x_t)^2}{\nu_2-2\nu_1 x_t+x_t^2} = 1-p$$

  this equation has a unique solution in the interval (t_1, t_2), and this solution is then the lower bound for $\mathrm{VaR}_p(X)$;
- for $p_2 < p < p_3$:

$$f_{low}^{(2)}(t) = 1 - p \;\Leftrightarrow\; \frac{(\nu_1-t)^2}{\nu_2-\nu_1^2+(\nu_1-m)(\nu_1-t)} = 1-p$$

  solving for t results in $t = \nu_1 - \frac{1}{2}(1-p)(\nu_1-m) - \frac{1}{2}\sqrt{(1-p)^2(\nu_1-m)^2+4(1-p)(\nu_2-\nu_1^2)}$, which is the lower bound for $\mathrm{VaR}_p(X)$;
- for $p_3 < p < 1$:

$$f_{low}^{(2)}(t) = 1 - p \;\Leftrightarrow\; \frac{\nu_2-t\nu_1}{b(b-m)} = 1-p$$

  solving for t results in $t = \frac{\nu_2-(1-p)b(b-m)}{\nu_1}$ which is the lower bound for $\mathrm{VaR}_p(X)$.

Now, in order to move from the results of Table 6 to those of Table 8, we note that if the support of the variable X is infinite, it is true that $b' = \nu_1$. This means that the switch point p_3 tends to one, and as a consequence, the last entry of Table 6 disappears.

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